International Journal of Theoretical Physics, Vol. 26, No. 7, 1987

Gauge Theories on Open Spin Space-Time Manifolds

Krystyna Bugajska[1]

Received February 8, 1987

We show that the presence of nontrivial gauge interactions can correspond to passing from some integrable to nonintegrable distributions transverse to the fibers of an appropriate principal G-bundle over M. In this way we obtain a whole family of new Lagrangians, one for each nontrivial element of $\mathrm{Hom}(\pi_1 M, Q)$. For the trivial map $\gamma \in \mathrm{Hom}(\pi_1 M, Q)$ we obtain the known Yang–Mills equations. In this way, for example, the "sectors" for electromagnetic interactions can correspond to the family of inequivalent spinor structures over M.

1. INTRODUCTION

In this paper we consider gauge theories from a slightly different point of view. Namely, we are especially interested in a horizontal distribution determined by a connection on an appropriate G-bundle P.

We assume that the presence of Yang–Mills interactions can be manifested by the map of a fiber transverse foliation of P given by a flat connection to a vector distribution given by horizontal subspaces. In this approach any "vacuum" state of the theory is related to a flat connection.

For any gauge theory we choose some discrete subgroup Q of the gauge group G. The group Q is given by the experimental data and contains elements that represent qualitative "charges" related to a given interaction. Now the set $\mathrm{Hom}(\pi_1 M, Q)$ numerates all "vacuum" states (or sectors of states) of our theory.

For the trivial map $\gamma \in \mathrm{Hom}(\pi_1 M, Q)$ we obtain the known Yang–Mills equations. For example, the Lagrangian of the theory will contain the usual Christoffel symbols and so on. But for nontrivial γ our Lagrangian should be built using other terms determined just by γ [see (27). For example, for a $U(1)$ gauge we have the following situation. For any standard Yang–Mills

[1]Institut für Theoretische Physik, der Technischen Universität Clausthal, D-3392 Clausthal-Zellerfeld, Germany.

field we have to take into account the whole family of Lagrangians [one for each $\gamma \in \mathrm{Hom}(\pi_1 M, Q)$], which can correspond to inequivalent spinor structures of (M, g).

This approach suggests that the homotopic properties of a space-time manifold cannot be trivial. The dual treatment of Yang–Mills fields (by horizontal distributions) also shows the manner in which we should relate topological properties of space-time to possible interactions.

In this paper we assume that a base manifold of P is an open spin space-time manifold. However, at the end we quote the vanishing theorem for flat bundles, which allows us to extend these considerations to other cases (which however, in our opinion may be less physical).

2. YANG–MILLS FIELDS

One of the major discoveries of current physics is the gauge theory (or Yang–Mills theory). The background for this theory is given by an appropriate principal G-bundle P over a semi-Riemannian space-time manifold M. The Yang–Mills field then is the curvature Ω_p of a connection ∇_p on P.

The action of G on P allows us to map each element X of the Lie algebra g of G into a vector field $\tilde{X}$ on P called a fundamental vector field. The set of all these vertical vector fields span the so-called vertical distribution V on P. [Let us recall that by a p-dimensional distribution on a manifold P ($p \leq \dim P$) we understand a function defined on P that assigns to each $n \in P$ a p dimensional linear subspace of the tangent space $T_n P$.]

A connection form on P is the g-valued differential 1-form ω, which assigns to each vertical vector $\tilde{X}$ a corresponding element $X \in g$ and is equivalent, i.e., satisfies the following condition:

$$R_a^* \omega(A) = Ada^{-1}\omega(A) \tag{1}$$

for every element $a \in G$ and for any vector field $A \in \Gamma(TP)$.

Dually we can tell that the connections of the principal bundle P is precisely the equivariant distribution H on P given by the kernel of ω, i.e.,

$$H_n = \mathrm{Ker}\,\omega_n \subset T_n P, \qquad \text{for every} \quad n \in P \tag{2}$$

and

$$H_{na} = R_a^* H_n \subset T_{na} P, \qquad \text{for every} \quad a \in G \tag{2'}$$

Moreover, we have

$$H \oplus V = TP \tag{3}$$

The horizontal distribution H is integrable if and only if a connection is flat. Namely, by the structure equation

$$\Omega_p = d\omega + \tfrac{1}{2}[\omega, \omega] \tag{4}$$

we have for horizontal vectors Y and Z

$$\begin{aligned}\Omega_p(Y, Z) = d\omega(Y, Z) &= Y\omega(Z) - Z\omega(Y) - \omega[Y, Z] \\ &= \omega[Y, Z]\end{aligned} \tag{5}$$

So the structure equation shows that for horizontal Y, Z

$$\Omega_p(Y, Z) = 0 \Leftrightarrow d\omega(Y, Z) = 0 \tag{6}$$

This means that H is involutive if and only if the condition (6) is satisfied, i.e., if and only if the curvature $\Omega_p = 0$. In this case the integrable distribution H defines a codimension q ($q = \dim G$) foliation $\mathcal{F}$ which is transverse to the fibres. The local submersions (see Appendix A) which determine this foliation are locally defined maps

$$f: P \to G \tag{7}$$

such that

$$\omega = f^*\theta \tag{8}$$

where θ is the canonical g-valued Mauer–Cartan form on G. The Maurer–Cartan equation $d\theta + \frac{1}{2}[\theta, \theta] = 0$ implies

$$d\omega + \tfrac{1}{2}[\omega, \omega] = 0 \tag{9}$$

for ω given by (8).

However, in the general case, especially in the cases considered by physicists, the connection is not flat. Yang–Mills fields do not identically vanish. Since the absence of Yang–Mills interactions can be related to a flat connection, and their presence to a horizontal nonintegrable distribution H, we can try to treat these interactions as some "forces" that destroy a foliation of P. In other words we will see that the presence of Yang–Mills interactions can be manifested by a map from an integrable horizontal distribution of some flat connection to H. Thus, first, we should know how many flat connections can exist on P and second, which of them can represent physical "basic" states.

It appears that the map from the canonical flat connection to H gives the known Yang–Mills equations. However, we also obtain other equations which correspond to maps from other concrete integrable distributions of flat connections (i.e., which correspond to other "basic" states of a theory). We will see that this is possible due to the following property of a flat bundle. Namely, if a G-bundle P admits a flat connection, then it can be treated as a bundle associated to the universal covering space $\tilde{M}$ of the space-time manifold M.

For this let us introduce the notion of the holonomy map h for any connection H on P. In a general case a connection on a bundle $P \to \pi M$ defines the map h from the loop space ΩM of M into G,

$$h : \Omega M \to G \tag{10}$$

Namely, let us take any loop σ at $x_0 \in M$. Let $\tilde{\sigma}$ be the unique horizontal lift of σ starting at an arbitrarily fixed point n_0 of the fiber $\pi^{-1}(x_0)$. The element $h(\sigma)$ is then the unique element in G such that $n_0 \cdot h(\sigma)$ is the endpoint of $\tilde{\sigma}$.

Let q be the canonical projection of loops to homotopy classes

$$q : \Omega M \to \pi_1 M \tag{11}$$

If the parallel transport depends only on the homotopy class of σ, then the holonomy map h factorizes through a representation

$$\gamma \colon \pi_1 M \to G \tag{12}$$

i.e., the diagram

$$\begin{array}{ccc} \Omega M & \longrightarrow & G \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle \gamma} & \\ & \pi_1 M & \end{array} \tag{13}$$

commutes. However, it is not a general case. If diagram (13) commutes, then our connection is flat. Moreover, this implies that we can reconstruct the bundle $P \to \pi M$ as follows.

Let $\tilde{M}$ be the universal convering of M, i.e., the principal bundle over M with a group $\pi_1 M$ acting properly discontinuous on $\tilde{M}$ (Kamber and Tandeur, 1975). The group $\pi_1 M$ acts on $\tilde{M} \times G$ by the covering transformation on the left factor and via γ on G.

The orbit space $\tilde{M} \times_{\pi_1} G$ inherits a right G action and there is a canonical bijection

$$\tilde{M} \times_{\pi_1} G \cong P \tag{14}$$

which is G-equivalent and hence a G-bundle isomorphism. In other words, we can tell that P can be given as a bundle associated to $\tilde{M}$ with fiber G and action of $\pi_1 M$ given by γ.

Since the action of $\pi_1 M$ preserves the product structure, so the product foliation of $\tilde{M} \times G$ arising from $\tilde{M} \times G \to G$ projects a foliation of P. This is just the foliation determined by a horizontal distribution of a flat (discrete holonomy group) connection. Thus, we see that if a bundle $P \to \pi M$ admits a flat connection, then any leaf of its horizontal distribution looks like a many-valued cross section of P. In addition, the projection π restricted to any leaf is a covering map of M. Moreover, the bundle P itself can be identified with $\tilde{M} \times_{\pi_1} G$ according to (14).

Now we have to consider the problem of the existence of a map from the integral horizontal distribution of a flat connection to the nonintegrable horizontal distribution H of some general connection on P. Because our space-time manifold is open, we can apply the general theory of foliation of open manifolds given, among others, by Haefliger (1971), Phillips (1968) and Gromov (1969) (see Appendix A).

According to their results (especially the Phillips-Gromov theorem), a continuous codimension-q plane field H on an open manifold P is homotopic to a smooth foliation if and only if the quotient vector bundle $\nu := TP/H$ is of foliated type. (This means that there exists a smooth codimension-q foliation of ν whose leaves are eveywhere transverse to the fibers.)

We see that the quotient bundle ν is canonically isomorphic to the vertical subbundle V of TP. Further, V is of foliated type iff P admits a flat connection.

In this way we come to the following situation in the gauge theory:

1. Yang-Mills fields define a nonintegrable horizontal distribution H on the total space of a principal G-bundle P (being the background for a gauge theory).
2. From the Phillips-Gromov theorem, H is homotopic to foliation if and only if P admits flat connection.
3. The presence of Yang-Mills interaction can be manifested by the map from a fiber transverse foliation of P (given by a flat connection) to a vector distribution (given by horizontal subspaces).

If we accept the above interpretation of the YM interactions, then it is natural to assume that any "vacuum" or "fundamental" state of a theory is related to a flat connection, i.e., flat G-bundle P. So the set of "basic" states could be given by the set of maps $\gamma \in \mathrm{Hom}(\pi_1 M, G)$. However, we suggest the following picture.

For any concrete gauge theory related to a concrete Lie group G we fix the image Q of γ's, i.e., we fix some discrete subset Q of elements of G. With each element of Q we associate a "qualitative charge" related to a given interaction.

Now, to any map $\gamma: \pi_1 M \to Q$ we relate a "vacuum" state or a sector of possible states of our theory.

For example, let $G = U(1)$, i.e., let P be the underlying space for the electromagnetic interaction. Because from experiments we have only two kinds of electric charges, $Q = Z_2$. Now the set of "basic" states or the set of sectors of physical states is given by

$$\mathrm{Hom}(\pi_1 M, Z_2) \tag{15}$$

Using the Scott and the Greenberg theorems (Whitson, 1973), we obtain that

$$\mathrm{Hom}(\pi_1 M, Z_2) \cong H^1(M, Z_2) \tag{16}$$

Further it is well known (Isham, 1978) that the set $H^1(M, Z_2)$ numerates inequivalent spinor structures on M. Thus, we see that the set of all inequivalent spinor structures on M can be the origin of the different "basic" states (or sectors) for electromagnetic interacting fields.

However, for this picture of YM fields we have to be sure that they are defined on a principal G-bundle which admits flat connection. We will see that this assumption does not introduce any physical restrictions for a reasonable physical theory based on an open space-time manifold.

It was said in the introduction that our space-time manifold M is an open spin manifold. This means, by the Geroch (1968) result, that M is parallelizable. Moreover, it implies that any $SU(n)$ principal bundle over M [or, more generally, and G-bundle with $G/SU(n)$ equivalent to a cell, for example, for $SL(2, \mathbb{C})$] has to be trivial (Isham, 1978) for $n \geq 2$.

In addition, the electromagnetic $U(1)$ bundle also has to be trivial when we accept its natural origin as, for example, given in Bugajska (1985).

In this way we see that principal bundles over a Lorentzian space-time manifold that are backgrounds for considered gauge theories admit flat connections.

Let us consider the problem of possible dynamical equations. Let P be some principal G-bundle as above with a fixed global trivialization $\sigma: M \to P$. Now any connection on $P \cong M \times G$ is completely determined by horizontal subspaces at points $M \times \{e\}$ (Kobayashi and Nomizu, 1963). Let $H_{(x,e)}$ be spanned by

$$H_{(x,e)} = \{e_i(x) + Y_i(x)\} \tag{17}$$

where $\{e_i(x)\}$ is a global field of orthonormal frames on M and $Y_i(x) \in$ Lie algebra g of G. Using the connection 1-form ω, we can write

$$\omega_{(x,e)}(e_i(x)) = -Y_i(x) \tag{18}$$

and

$$\omega_{(x,a)}(e_i(x)) = -ada^{-1} Y_i(x), \qquad \forall a \in G \tag{19}$$

So the 1-form of a connection is determined by a g-valued 1-form, say w, on M

$$w = \sigma^* \omega \tag{20}$$

and

$$H_{(x,e)} = \{e_i(x) - w(e_i(x))\} \tag{21}$$

Let us consider another connection ω' on P which is induced by such a 1-form w' that

$$w'(e_i(x)) = -w(e_i(x)) \tag{22}$$

Then

$$\omega'_{(x,a)}(e_i(x)) = -\omega_{(x,a)}(e_i(x)), \qquad \forall x \in M, \quad \forall a \in G \tag{23}$$

We can easily check that for any vector field A on P

$$A(x, a) = \sum_i A^i(x, a)e_i(x) + \sum_\alpha A^\alpha(x, a)E_\alpha \tag{24}$$

where E_α are generators of G, we have

$$\tfrac{1}{2}(\omega + \omega')A(x, a) = A^\alpha(x, a)E_\alpha \tag{25}$$

This means that

$$\tfrac{1}{2}(\omega + \omega') = p^*\theta \tag{26}$$

where $p: M \times G \to G$. (We recall that a gobal trivialization of $P \cong M \times G$ is given by a global section σ.) By (23), we see that the Christoffel symbols of connections ω and w' are equal up to sign. From (26) we obtain that $\frac{1}{2}(\omega + \omega')$ is the canonical flat connection on the trivial bundle P. It corresponds to a foliation, or equivalently, to a basic state given by the trivial map $\gamma \in \mathrm{Hom}(\pi_1 M, Q)$.

In our approach we understand the presence of Yang-Mills interactions as some "forces" that destroy a foliation of P (i.e., change some "basic" noninteracting state to an interacting one). However, we should consider the whole family of foliations of P (i.e., "basic" states) given by the set $\mathrm{Hom}(\pi, M, Q)$. But how can these nontrivial "basic" states be incorporated into a theory? We can see that to any connection ω' on P and to any "basic" state $\gamma \in \mathrm{Hom}(\pi_1 M, Q)$ we can relate a unique connection ω such that locally

$$\tfrac{1}{2}(\omega + \omega') = f^*\theta \tag{27}$$

For this we have to show that local functions f: $P \to G$ are determined by γ. Let us recall that any element γ defines P as $\tilde{M} \times_{\pi_1} G$. In other words, every element of P can be treated as a class

$$[\tilde{m}, a]_\gamma = (\tilde{m}, a)\pi_1 M = [\tilde{m}\rho, \gamma(\rho^{-1})a]_\gamma, \qquad \forall_\rho \in \pi_1 M \tag{28}$$

Let $\{U_\alpha\}$ be a contractible open covering of M. Let $\{\tilde{m}_\alpha\}$ be a local trivialization of $\tilde{M}$ related with $\{U_\alpha\}$. From (28) we see that a local map $f_\alpha^\gamma: P \to G$

$$f_\alpha^\gamma[m_\alpha, a]_\gamma = a \tag{29}$$

is different for different γ as well as that a set $\{f_\alpha^\gamma\}$ determines a concrete foliation of P by pullback of the Maurer–Cartan form.

In this way we have obtained that for the trivial map γ we have known equations [i.e., ω in (26) is just the connection considered by physicists], but besides we should consider the whole class of dynamical equations with ω given by (27) and (29). For example, for $U(1)$ gauge it corresponds to the inclusion of the whole family of inequivalent spinor structures in to a theory.

This approach suggests that the homotopic properties of a space-time manifold cannot be trivial. Also this approach shows in which manner we should relate topological properties of space-time with possible interactions. Namely, the qualitative "charges" given by a concrete interaction have to be obtained as an image of $\pi_1 M$. If M is simply connected, then any principal G-bundle that admits a flat connection has to be trivial and the flat connection has to be isomorphic with the canonical one (Kobayashi and Nomizu, 1963). In this case we have only one "basic" state and only standard equations.

It seems that this interpretation of Yang–Mills fields as well as the definition of "basic" states as a foliation of P (or flat connection) can be generalized to compact manifolds and the larger family of Lie groups. But for this we should describe more precisely the necessary condition for a flatness principal bundle (see Appendix B).

APPENDIX A

Let N and N' be two manifolds of respective dimensions n and n'; $n \geq n'$. A map $\pi: N \to N'$ is called submersion if

$$d\pi_x : T_x N \to T_{\pi(x)} N'$$

is surjective $\forall x \in N$.

A codimension-q foliation $\mathscr{F}$ of N is a decomposition of N into a union of disjoint connected codimension-q submanifolds $N = \bigcup_{\mathscr{L} \in \mathscr{F}} \mathscr{L}$ called the leaves of foliation. Moreover, for each $x \in N$ there is a neighborhood U of x and a smooth submersion $\pi_U: U \to R^q$ such that $\pi_U^{-1}(y)$, $y \in R^q$, is a leaf of $\mathscr{F}|_U$, the restriction of the foliation to U.

Let N be a manifold with a smooth, codimension-q foliation $\mathscr{F}$ defined by local submersions $\pi_U: U_\mu \to R^q$ (here $\{U_\mu\}$ is an open covering of N).

For any manifold P we define the space Trans(P, $N_{\mathcal{F}}$) of smooth maps that are transverse to $\mathcal{F}$. (We recall that a mapping $f: P \to N$ is said to be transverse to $\mathcal{F}$ provided each of the compositions $\pi_\mu \circ f$ is a submersion of $f^{-1}(U_\mu)$ into R^q.) Of course, for $f \in$ Trans(P, $N_{\mathcal{F}}$), $f^*\mathcal{F}$ given by the local submersions

$$\pi_\mu \circ f: f^{-1}(U_\mu) \to R^q \tag{A1}$$

is a codimension-q foliation of P.

Now let us consider the space Epi(TP, $\nu\mathcal{F}$)) of continuous bundle maps from the tangent bundle TP of P to the normal bundle $\nu(\mathcal{F})$ of the foliation, which is an epimorphism on each fiber. [We recall that the collection of spaces tangent to the leaves of foliation is called the tangent bundle of the foliation and is usually denoted by $\tau(\mathcal{F})$. The quotient bundle $TN/\tau(\mathcal{F})$ is called the normal bundle of foliation and is usually denoted by $\nu(\mathcal{F})$.]

There is a natural continuous map

$$\chi: \mathrm{Trans}(P, N_{\mathcal{F}}) \to \mathrm{Epi}(TP, \nu(\mathcal{F})) \tag{A2}$$

given by

$$\chi(f) = p \circ df \tag{A3}$$

where $df: TP \to TN$ is a differential of f and $p: TN \to \nu(\mathcal{F})$ is the projection. The space Epi is considerably larger than Trans. The starting fact for open manifolds is the following theorem (Lawson, 1974).

Theorem (Gromov–Phillips). For any open manifold P the map χ in (A2) is a weak homotopy equivalence, that is, it induces isomorphisms on all homotopy and homology groups.

From this theorem we obtain that in particular χ establishes a one-to-one correspondence between the connected component, i.e., π_0 of these two spaces. Thus we conclude that every codimension-q plane field D that can be written as $D = \mathrm{Ker}\,\beta$ for some $\beta \in$ Epi(TP, $\nu(\mathcal{F})$) is a homotopic to a foliation.

Now we have the following problem: Given a horizontal plane H on a principal bundle P determined by a 1-form of connection ω, is it homotopic to a transverse to the fibers of P foliation? In other words, can we find N, $\mathcal{F}$, and β so that the relation

$$H = \mathrm{Ker}\,\beta \tag{A4}$$

To solve this problem, let us follow an idea of Milnor (1957). He has defined N in a canonical way as the total space of the quotient vector bundle $\nu = TP/H$. In our case ν is isomorphic to the vertical subbundle V of TP [see (3)].

We say that a q-dimensional vector bundle $V \to M$ is of foliated type if there exists a smooth codimension-q foliation of V whose leaves are everywhere transverse to the fibers. Then for any leaf of this foliation the projection into M is a local diffeomorphism.

Now we have the following lemma (Lawson, 1974).

Lemma. Let P be an open manifold. Then a continuous codimension-plane field H on P is homotopic to a smooth foliation if and only if the quotient vector bundle $\nu = TP/H$ is of foliated type.

APPENDIX B

In a general case the flatness of a bundle can be expressed in purely topological terms. For this let us look at the characteristic classes of flat bundles.

Let BG be the classifying space of G (Husemoller, 1966), i.e., any principal G-bundle P over M can be induced from the universal bundle over BG by a classifying map

$$\xi : M \to BG \tag{B1}$$

(ξ is determined by P up to homotopy.) The universal covering bundle $\tilde{M}$ of M is classified by a map

$$\zeta : M \to B\pi_1 M \tag{B2}$$

Now P is flat if and only if there exists a homomorphism $\gamma : \pi_1 M \to G$ such that the diagram

$$\begin{array}{ccc} B\pi_1 M & \xrightarrow{B\gamma} & BG \\ & {\scriptstyle \zeta}\searrow \quad \swarrow{\scriptstyle \xi} & \\ & M & \end{array} \tag{B3}$$

commutes up to homotopy (Milnor, 1957). Again we see that if M is simply connected, then $B\pi_1 M$ is contractible and P has to be a trivial bundle.

Diagram (B3) induces a commutative diagram of cohomology groups

$$\begin{array}{ccc} H^*(B\pi_1 M) & \xleftarrow{(B\gamma)^*} & H^* BG \\ & {\scriptstyle \zeta *}\searrow \quad \swarrow{\scriptstyle \xi *} & \\ & H^* M & \end{array} \tag{B4}$$

It can be seen that $H^*(B\pi_1 M) \cong H^*(\pi_1 M)$. Now, because ξ^* factorizes through $H^*(B\pi_1 M)$, the cohomological properties of the fundamental group $\pi_1 M$, reflected in ξ^*. For example, if $\pi_1 M$ is finite, then the rational cohomology group $H^i(\pi_1 M, Q) = 0$ for $i > 0$. Thus, the rational characteristic classes of P are in this case trivial. In a general case we have the following vanishing theorem (Kamber and Tonleur (1968).

Theorem. Let M be a CW complex and P a flat G-bundle on M. Suppose G has finitely many path-connected components and is either a compact or a complex and reductive Lie group. Then the characteristic homomorphism

$$\xi^* : H^*(BG, R) \to H^*(M, R) \tag{B5}$$

is trivial, i.e., zero in positive degrees. [The coefficient field in (B5) may be replaced by any field of characteristic zero.]

If G is compact, then this result follows from the Chern-Weil theorem (Pittie, 19), representing the characteristic classes of P by polynomials in the curvature form of a connection in P.

For various other classes of Lie groups G there are examples of flat G-bundles with nontrivial real characteristic classes. These characteristic classes cannot be determined by the curvature form of any connection in P; there is also no Chern-Weil theorem in these cases.

ACKNOWLEDGMENT

I thank Prof. H. Doebner for his hospitality at Clausthal University as well as for his help and many valuable discussions.

REFERENCES

Bugajska, K. (1985). *Journal of Mathematical Physics*, **26**, 2909.

Geroch, R. (1968). *Journal of Mathematical Physics*, **9**, 1739.

Gromow, M. (1969). *Izvestiya Akadremii Nauk SSSR Seriya Matematicheskoi*, **33**, 707-734.

Haefliger, A. (1971). In *Lecture notes in mathematics*, Vol. 209, Springer, New York.

Husemoller, D. (1966). *Fibre bundles*, McGraw-Hill, New York.

Isham, C. (1978). *Proceedings of the Royal Society of London A*, **364**, 591-599.

Kamber, K., and Tondeur, P. (1968). *Flat manifolds*, In *Lecture notes in mathematics*, Vol. 67, Springer, New York.

Kamber, F., and Tondeur, P. (1975). Foliated bundles and characteristic classes, in *Lecture notes in mathematics*, Vol. 493, Springer, New York.

Kobayashi, S., and Nomizu, K. (1963). *Foundation of differential geometry*, Vol. I, Interscience, New York.

Lawson, B. (1974). *Bulletin of the American Mathematical Society*, **80**, 369-418.

Milnor, J. (1975). *Commentarii Mathematici Helvetici*, **32**, 215-223.

Phillips, A. (1968). *Commentarii Mathematici Helvetici*, **43**, 204-211.

Pittie, H. (1970). *Characteristic classes of foliations*, In *Research notes in mathematics*, Vol. 10, Pitman, New York.